ARKIV FÖR FYSIK Band 19 nr 2

Communicated 15 September 1960 by O. Klein

Expansions of wave functions in terms of Bessel functions

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Summary

Certain solutions of wave equations are expanded in a series containing an arbitrary parameter. Depending on the choice of the parameter different expansions in terms of Bessel functions and special cases thereof are obtained. The general expansion coefficient and a recurrence relation for their calculation is given. The expansions can be used, e.g., to study the asymptotic behaviour of transition matrix elements of quantum mechanics. Many previously known expansions can be obtained as special cases.

In agreement with methods used in a previous paper dealing with Coulomb wave functions [1], we will introduce the function

$$\Phi_2(b, b'; c, xr, yr) = \sum_{\substack{n=0\\m=0}}^{\infty} \frac{(b)_m (b')_n (xr)^m (yr)^n}{(c)_{m+n}\, m!\, n!}, \tag{1}$$

where a symbol such as $(b)_m$ means

$$(b)_m = b(b+1)(b+2)(b+3)\ \ldots\ (b+m-1).$$

When $b+b'=c$ this function reduces to [1], [2],

$$\Phi_2(b, b'; b+b'; xr, yr) = e^{yr}\, {}_1F_1[b; b+b'; (x-y)r], \tag{2}$$

where the confluent hypergeometric function on the right-hand side is defined by

$${}_1F_1(a; b; z) = \sum_{n=0}^{\infty} \frac{(a)_n z^n}{(b)_n n!}. \tag{3}$$

Thus we may in many cases express the solutions of wave equations in terms of the function (1) by appropriate choice of the parameters.

Henceforth we will work with the function (1). The restriction $b+b'=c$ will not be used since it does not simplify the subsequent derivations.

The function (1) can be expanded in a series

$$\Phi_2(b, b'; c; xr, yr) = \sum_{n=0}^{\infty} \frac{a_n r^n}{(c)_n}\, {}_0F_1(c+n; -zr), \tag{4}$$

where z is an arbitrary parameter, which of course is also contained in the expansion coefficients a_n. The ${}_0F_1$ function is defined by

$$ {}_0F_1(a; z) = \sum_{n=0}^{\infty} \frac{z^n}{(a)_n\, n!}. \tag{5} $$

This function is related to the Bessel function by

$$ J_c(z) = \frac{(z/2)^c}{\Gamma(c+1)}\, {}_0F_1\left(c+1;\ -\frac{z^2}{4}\right) $$

but the function (5) will be preferred here since it simplifies the notation.

The expansion (4) was derived in order to study the asymptotic behaviour of the Φ_2 function in a case where $|b|$ and $|b'| \to \infty$ and $|x|$ and $|y| \to 0$. The general expression for the expansion coefficients has been expressed in terms of a known function in a previous paper [1]

$$ a_n = \frac{(b')_n\, y^n}{n!}\, \Phi_1\left(-n, b;\ 1-b'-n;\ \frac{x}{y}, \frac{z}{y}\right). \tag{6} $$

The Φ_1 function is one of the confluent Appell functions [3]

$$ \Phi_1(a, b;\ c;\ x, y) = \sum_{\substack{m=0\\ n=0}}^{\infty} \frac{(a)_{m+n}\,(b)_m\, x^m\, y^n}{(c)_{m+n}\, m!\, n!}. \tag{7} $$

In this paper we will derive a recursion formula for the coefficients a_n. This is considerably simplified by expressing a_n in terms of the function Φ_1. The recursion relation can be obtained from a differential equation satisfied by the Appell function

$$ F_1(a, b, b';\ c;\ x, y) = \sum_{\substack{m=0\\ n=0}}^{\infty} \frac{(a)_{m+n}\,(b)_m\,(b')_n\, x^m\, y^n}{(c)_{m+n}\, m!\, n!}. \tag{8} $$

Having derived an appropriate relation between the F_1 functions from the differential equation the desired relation between the Φ_1 functions can be obtained by confluence

$$ \lim_{\varepsilon \to 0} F_1(a, b, 1/\varepsilon;\ c;\ x, \varepsilon y) = \Phi_1(a, b;\ c;\ x, y). \tag{9} $$

Considering $F_1(a, b, b';\ c;\ x, y)$ as a function of y it satisfies the following differential equation [4]

$$ \left[B_0 \frac{d^3}{dy^3} + B_1 \frac{d^2}{dy^2} + B_2 \frac{d}{dy} + B_3\right] F_1(a, b, b';\ c;\ x, y) = 0, \tag{10} $$

where

$$
\begin{aligned}
&B_0 = y(1-y)(x-y),\\
&B_1 = (c+1-b)(1-y)(x-y) + (c-a-b'-2)\,y(x-y) - (b+b'+1)\,y(1-y),\\
&B_2 = (b'+1)\,[(a+b+b'+1-c)\,y - c(1-y) + (b-a-1)(x-y)],\\
&B_3 = ab'(b'+1).
\end{aligned}
$$

Inserting the well-known expression for the derivatives

$$\frac{d^n}{dy^n} F_1(a, b, b'; c; x, y) = \frac{(a)_n\,(b')_n}{(c)_n} F_1(a+n, b, b'+n; c+n; x, y)$$

into (10) we obtain a relation between four F_1 functions, in which all parameters except b are increased by one unit. Next introduce $b' \to 1/\varepsilon$, $y \to \varepsilon y$ and carry out the limit $\varepsilon \to 0$, whereby the F_1 function goes over to the desired Φ_1 functions according to (9). We obtain

$$
\begin{aligned}
&(a+1)(a+2)\,xy\,\Phi_1(a+3, b; c+3, xy)\\
&\qquad + (a+1)(c+2)\,[(c-b+1)\,x - y - xy]\,\Phi_1(a+2, b; c+2; x, y)\\
&\qquad + (c+1)(c+2)\,[y - c + (b-a-1)\,x]\,\Phi_1(a+1, b; c+1; x, y)\\
&\qquad + c(c+1)(c+2)\,\Phi_1(a, b; c; x, y) = 0.
\end{aligned}
$$

Introducing a_n according to (6) we obtain

$$
\begin{aligned}
&xyz\,a_{n-2} + [(1-b-b'-n)\,xy - xz - yz]\,a_{n-1}\\
&\qquad + [z + (n+b)\,x + (n+b')\,y]\,a_n - (n+1)\,a_{n+1} = 0,
\end{aligned}
$$

which is the desired recurrence relation.

The coefficients a_n are defined from the expansion [1]

$$e^{-zr}(1-xr)^{-b}(1-yr)^{-b'} = \sum_{n=0}^{\infty} a_n r^n$$

and this expansion is convergent when

$$|r| < \frac{1}{|x|}, \quad |r| < \frac{1}{|y|}.$$

Since ${}_0F_1(c+n; -zr) = 0\,(1)$ for large values of n, the expansion (4) converges as

$$\sum_{n=0}^{\infty} \frac{a_n}{(c)_n} r^n,$$

which is obviously convergent for all values of r and for any values of the parameters.

Special cases of the expansion derived here have been derived and investigated by many authors among which we mention Tricomi [5], Abramowitz [6] and Breit & Hull Jr. [7]. A numerical investigation of an expansion of the nonrelativistic Coulomb wave functions in terms of modified Bessel functions has been carried out by Fröberg [8].

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Tryckt den 27 januari 1961

Uppsala 1961. Almqvist & Wiksells Boktryckeri AB